

BOUNDARY SOLUTIONS OF THE QUANTUM YANG-BAXTER EQUATION AND SOLUTIONS IN THREE DIMENSIONS

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ABSTRACT. Boundary solutions to the quantum Yang–Baxter (qYB) equation are defined to be those in the boundary of (but not in) the variety of solutions to the “modified” qYB equation, the latter being analogous to the modified classical Yang–Baxter (cYB) equation. We construct, for a large class of solutions r to the modified cYB equation, explicit “boundary quantizations”, i.e., boundary solutions to the qYB equation of the form $I + tr + t^2 r_2 + \dots$. In the last section we list and give quantizations for all classical r -matrices in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$.

1. INTRODUCTION

Solutions to the quantum Yang–Baxter (qYB) equation, so-called “ R -matrices,” have many applications in mathematical physics and are essential to the theory of quantum groups, but explicit solutions are scarce. Finding directly all constant $n^2 \times n^2$ matrix solutions reduces to solving n^6 homogeneous cubic equations in n^4 variables, a task feasible only in the case $n = 2$, see [5]. Usually the search is restricted to one for those R -matrices which are quantizations (in the sense below) of “ r -matrices”, i.e., solutions to the classical Yang–Baxter (cYB) equation, but even these R are somewhat elusive. The problem is two-fold – first find all r -matrices, then quantize them. In [2], we introduced the notion of a boundary solution to the classical Yang–Baxter equation in the hope that there may be some reasonable classification for these. Section 2 introduces the natural extension of this concept to that of a boundary solution to the quantum Yang–Baxter equation. For a large class of boundary classical r -matrices we explicitly construct in Section 3 associated boundary quantizations. Finally, Section 4 contains a complete classification (up to conjugacy) of all R -matrices which are quantizations of classical r -matrices in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$. While not all of these R are constructed as boundary solutions, in a sense the most interesting ones are. It remains an open question to determine precisely which ones actually are boundary R -matrices.

2. THE QUANTUM YANG-BAXTER EQUATIONS

Let V be an n -dimensional vector space over a field k . A quantum R -matrix is an operator $R : V \otimes V \rightarrow V \otimes V$ satisfying the quantum Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

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where as usual R_{ij} denotes R operating in the i and j factors of the triple tensor product $V^{\otimes 3} = V \otimes V \otimes V$. We define the *modified* qYB equation (MQYBE) to be

$$R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = \lambda((123)R_{12} - (213)R_{23})$$

where λ is a non-zero element of the coefficient field and (123) , (213) denote the corresponding permutations of the tensor factors of $V^{\otimes 3}$. Correspondingly, we call R a modified R -matrix if it satisfies the MQYBE. A solution to either equation is *unitary* if $R_{21} = R^{-1}$.

Remark. Modified R -matrices are closely connected to Hecke symmetries, for if R is a modified unitary R -matrix then $((12)R - \lambda I)/\sqrt{(1 - \lambda^2)}$ satisfies the braid and Hecke relations with $q = (1 - \lambda)/\sqrt{1 - \lambda^2}$, see [3] for a detailed discussion.

A natural problem is to determine the structure of the spaces of solutions to both the original and modified quantum Yang-Baxter equations. It is clear that if R satisfies either then so does any scalar multiple of R , so it is meaningful to view $R \in \mathbb{P}(M_{n^2}(k))$. While little is known in general, the following simple result is helpful in determining many solutions.

Theorem 2.1. *Let \mathfrak{R} , respectively \mathfrak{R}' , denote the subsets of $\mathbb{P}(M_{n^2}(k))$ consisting of solutions to the quantum Yang-Baxter equation, respectively modified quantum Yang-Baxter equation, and let $\overline{\mathfrak{R}'}$ be the Zariski closure of \mathfrak{R}' . Then*

1. \mathfrak{R} is a closed variety.
2. $\overline{\mathfrak{R}'} - \mathfrak{R}' \subset \mathfrak{R}$.
3. \mathfrak{R}' is quasi-projective, i.e. it is an open subset of its closure.

Proof.

1. Each entry of $R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12}$ is a homogeneous cubic polynomial in the entries of R and \mathfrak{R} , being the locus of common zeroes of these polynomials, is therefore a closed variety.
2. Define maps $\phi : (\mathbb{P}(M_{n^2}(k)) - \mathfrak{R}) \rightarrow \mathbb{P}(M_{n^3}(k))$ with $\phi(R) = R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12}$ and $\psi : (\mathbb{P}(M_{n^2}(k)) - I) \rightarrow \mathbb{P}(M_{n^3}(k))$ with $\psi(R) = ((123)R_{12} - (213)R_{23})$. Since $\mathfrak{R}' = \phi^{-1}(\text{Im}(\phi) \cap \text{Im}(\psi))$ and $\text{Im}(\phi) \cap \text{Im}(\psi)$ is closed, it follows that \mathfrak{R}' is closed in $\mathbb{P}(M_{n^2}(k)) - \mathfrak{R}$. Thus $\overline{\mathfrak{R}'} - \mathfrak{R}' \subset \mathfrak{R}$.
3. To show that \mathfrak{R}' is open in $\overline{\mathfrak{R}'}$, it suffices to verify that $\overline{\mathfrak{R}'} - \mathfrak{R}'$ is closed in $\overline{\mathfrak{R}'}$. Since $\overline{\mathfrak{R}'} - \mathfrak{R}' = \overline{\mathfrak{R}'} \cap \mathfrak{R}$ and both $\overline{\mathfrak{R}'}$ and \mathfrak{R} are closed, it follows that $\overline{\mathfrak{R}'} - \mathfrak{R}'$ must also be closed.

□

In light of Theorem 2.1, we call an $R \in \overline{\mathfrak{R}'} - \mathfrak{R}'$ a *boundary solution* to the QYBE or boundary R -matrix.

3. CONSTRUCTION OF BOUNDARY SOLUTIONS

In [2] we introduced boundary solutions of the classical Yang-Baxter equation and gave some procedures for constructing them. In this section we “quantize” some of those.

Suppose that R is a modified unitary quantum R -matrix of the form $R = I + tr + O(t^2)$. Then the “(semi)classical limit” r is a skew modified classical r -matrix, that is, $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$ is a non-zero $\mathfrak{sl}(n)$ -invariant. Similarly, the classical limit of a quantum unitary R -matrix is a skew classical r -matrix. In either

case we call R a “quantization” of r . Now let $r \in \mathfrak{sl}(n) \wedge \mathfrak{sl}(n)$ be a modified r -matrix, and suppose that R is a modified unitary R -matrix with classical limit r . Let \mathcal{O}_r and \mathcal{O}_R denote, respectively, the $SL(n)$ -orbits of r and R and let $\overline{\mathcal{O}}_r$ and $\overline{\mathcal{O}}_R$ be their closures. For any $\tilde{r} \in \overline{\mathcal{O}}_r - \mathcal{O}_r$, we seek an $\tilde{R} \in \overline{\mathcal{O}}_R - \mathcal{O}_R$ which quantizes \tilde{r} . In [2] it was shown that such an \tilde{r} is a boundary classical r -matrix so what we are really looking for is an associated boundary R -matrix. There is presently no complete answer to this question, although a conjectured explicit form for the quantization of all modified classical r does exist, see [3]. (It is known that every such r does admit a quantization.) We can however give a positive answer in many cases.

Remark. Given an r , the quantizations we produce will be of the form $R = 1 + 2tr + O(t^2)$ which is, course, equivalent to finding an R with classical limit r . This will be particularly useful in the next section as we can eliminate powers of $1/2$ in the R -matrices.

Set $\gamma = \sum_{i < j} e_{ij} \wedge e_{ji}$, this is commonly called the Drinfel’d-Jimbo modified classical r -matrix since it is the classical limit of the standard quantization of $SL(n)$. It was shown in [3] that the appropriate quantization of γ is $\exp(2t\gamma)$. This is a modified unitary quantum R -matrix and the corresponding Hecke symmetry coincides with that of the classic R -matrix associated with $\mathcal{O}_q(SL(n))$.

Theorem 3.1. *Let $\gamma' \in \overline{\mathcal{O}}_\gamma - \mathcal{O}_\gamma$. Then $\exp(2t\gamma')$ is a boundary solution of the quantum Yang-Baxter equation.*

Proof. Since $\gamma' \in \overline{\mathcal{O}}_\gamma$, it follows that $\exp(2t\gamma') \in \overline{\mathcal{O}}_{\exp(2t\gamma)} \subset \overline{\mathfrak{R}}'$. Now because γ' is a classical r -matrix, $\exp(2t\gamma') \notin \mathfrak{R}'$ and so by Theorem 2.1 it follows that $\exp(2t\gamma')$ is a boundary solution to the quantum Yang-Baxter equation. \square

As is well known, $\gamma + \beta$ is a modified r -matrix for any $\beta \in \mathfrak{h} \wedge \mathfrak{h}$. These are the classical limits of the standard multi-parameter R -matrices. The modified quantum R -matrix which quantizes $\gamma + \beta$ is $\exp(t\beta) \cdot \exp(2t\gamma) \cdot \exp(t\beta)$, see [3]. For a large class of boundary classical r -matrices in $\overline{\mathcal{O}}_{\gamma+\beta} - \mathcal{O}_{\gamma+\beta}$ the same type quantization is possible.

Theorem 3.2. *Suppose $x \in \mathfrak{sl}(n)$ and suppose*

$$\exp(vad x)(\gamma + \beta) = (\gamma + \beta) + v(\gamma_1 + \beta_1) + \cdots + v^d(\gamma_d + \beta_d).$$

Then $\gamma_d + \beta_d \in \overline{\mathcal{O}}_{\gamma+\beta} - \mathcal{O}_{\gamma+\beta}$ and $\exp(t\beta_d) \cdot \exp(2t\gamma_d) \cdot \exp(t\beta_d)$ is a boundary R -matrix which quantizes $\gamma_d + \beta_d$.

Proof. According to [2], $\gamma_d + \beta_d \in \overline{\mathcal{O}}_{\gamma+\beta} - \mathcal{O}_{\gamma+\beta}$ and so is a boundary classical r -matrix. Set $M = \exp(t\beta) \cdot \exp(2t\gamma) \cdot \exp(t\beta)$. Then $\exp(t\beta_d) \cdot \exp(2t\gamma_d) \cdot \exp(t\beta_d) \in \overline{\mathcal{O}}_M \subset \overline{\mathfrak{R}}'$. But since its classical limit is an r -matrix, $\exp(t\beta_d) \cdot \exp(2t\gamma_d) \cdot \exp(t\beta_d) \notin \mathfrak{R}'$ and so by Theorem 2.1 it must be a boundary solution to the quantum Yang-Baxter equation. \square

4. QUANTIZATION OF THE CLASSICAL r -MATRICES IN $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$

We now consider the quantizations of all classical r -matrices in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$. The quantizations of the modified classical r -matrices in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$ are well known. Up

to automorphism of $\mathfrak{sl}(3)$, there are two distinct types of such r , the one-parameter standard family

$$r_\lambda = e_{12} \wedge e_{21} + e_{13} \wedge e_{31} + e_{23} \wedge e_{32} + \lambda(e_{11} - e_{22}) \wedge (e_{22} - e_{33})$$

and what we have called the Cremmer-Gervais solution

$$r_{CG} = r_{1/3} + 2e_{12} \wedge e_{32}.$$

The modified R -matrix associated with r_λ was discussed in the previous section and the Cremmer-Gervais modified R is

$$q^{-\beta} \left\{ \exp(2t\gamma) + 2 \sin(t)(q^{1/2}e_{12} \otimes e_{32} - q^{-1/2}e_{32} \otimes e_{12}) \right\} q^{-\beta}$$

with $\beta = (1/3)(e_{11} - e_{22}) \wedge (e_{22} - e_{33})$ and $q = \sec(t) - \tan(t)$, see [3].

Now let us focus on the classical r -matrices in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$. Using the homological interpretation (due to Drinfel'd) of the classical Yang-Baxter equation, Stolin [6] has given a description of classical r -matrices in terms of “quasi-Frobenius” Lie algebras. A Lie algebra \mathfrak{f} is quasi-Frobenius if there is a non-degenerate map $\phi : \mathfrak{f} \wedge \mathfrak{f} \rightarrow k$. The inverse matrix of ϕ is then a classical r -matrix. Now if \mathfrak{g} is a simple Lie algebra, and $r \in \mathfrak{g} \wedge \mathfrak{g}$ is a classical r -matrix, then there is a unique proper subalgebra \mathfrak{f} of \mathfrak{g} for which $r \in \mathfrak{f} \wedge \mathfrak{f}$ and is non-degenerate. Call \mathfrak{f} the *carrier* of r . Exploiting the relatively small dimension, Stolin gave a complete list of quasi-Frobenius subalgebras of $\mathfrak{sl}(3)$ and computed enough of their cohomology to determine the number of classical r -matrices each carries up to automorphism. However, no explicit form of these r nor of their quantizations was given in [6].

The rest of this note is devoted to listing explicitly all the classical r -matrices in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$ and finding unitary quantizations of each. If \mathfrak{f} is quasi-Frobenius then $r(\mathfrak{f})$ will denote a classical r -matrix carried by \mathfrak{f} and $R(\mathfrak{f})$ will be an R -matrix which quantizes $r(\mathfrak{f})$.

Not all r -matrices in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$ seem to be boundary solutions to the classical Yang-Baxter equation so other techniques besides those in Theorems 3.1 and 3.2 are needed to quantize them. The following theorem from [4] is useful here.

Theorem 4.1. *Let \mathfrak{s} be the two-dimensional Lie algebra generated by H and E with relation $[H, E] = E$. Set $H^{(d)} = H(H+1)(H+2) + \cdots (H+(d-1))$. Let $F_m = \sum_{i=0}^m (-1)^i \binom{m}{i} E^{m-i} H^{(i)} \otimes E^i H^{(m-i)}$ and let $F = \sum_{i=0}^\infty (t^i/i!) F_m$. Then $F_{21}^{-1} F$ is a universal quantization of $2(E \wedge H)$.*

Specializing the universal R to any finite dimensional representation of \mathfrak{s} gives an R -matrix. Another technique which sometimes quantizes non-boundary r -matrices is to simply exponentiate them. We are not sure which class of r -matrices has the property that the exponential map provides an R -matrix, but, fortunately, for those r in $\mathfrak{sl}(3) \wedge \mathfrak{sl}(3)$ which are not covered by Theorems 3.1, 3.2, or 4.1 it turns out that the exponential map works.

Up to automorphism, there are 10 classes of quasi-Frobenius subalgebras of $\mathfrak{sl}(3)$. Their dimensions must be even. We will for the most part follow the numbering and notation of these Lie algebras used in [1], Section 3.1.

- (i) The only six dimensional subalgebra of $\mathfrak{sl}(3)$ is

$$\mathfrak{p} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

and $r(\mathfrak{p}) = (2e_{11} - e_{22} - e_{33}) \wedge e_{12} + (e_{11} + e_{22} - 2e_{33}) \wedge e_{23} + 3e_{13} \wedge e_{32}$ is the unique classical r with carrier \mathfrak{p} , see [2]. Let Q be the Cremmer-Gervais modified R . Let $x = -e_{12} - (1/2)e_{23}$ and set $R(\mathfrak{p}) = \lim_{t \rightarrow 0} \exp\left(\frac{3m}{t} \text{ad}(x)\right) \cdot Q$. Then

$$R(\mathfrak{p}) = \begin{pmatrix} 1 & 2m & m^2 & -2m & 2m^2 & m^3 & m^2 & -m^3 & -2m^4 \\ 0 & 1 & m & 0 & m & m^2 & 0 & m^2 & -2m^3 \\ 0 & 0 & 1 & 0 & 0 & m & 0 & 3m & -2m^2 \\ 0 & 0 & 0 & 1 & -m & m^2 & -m & m^2 & 2m^3 \\ 0 & 0 & 0 & 0 & 1 & m & 0 & -m & 2m^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2m \\ 0 & 0 & 0 & 0 & 0 & -3m & 1 & -m & -2m^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is a boundary R -matrix which quantizes $r(\mathfrak{p})$.

(ii) The four dimensional subalgebra

$$\mathfrak{r} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

carries a one-parameter family of classical r -matrices. If $x = (3a/2)e_{12} + (3/2)e_{13}$ then $\exp(t \text{ad } x) \cdot r_{1/3} = r_{1/3} + t[x, r_{1/3}]$ and so $[x, r_{1/3}] = d_1 \wedge e_{12} + d_2 \wedge e_{13}$ where $d_1 = a(-e_{11} + 2e_{22} - e_{33}) + 3e_{23}$ and $d_2 = -2e_{11} + e_{22} + e_{33}$ is a boundary r -matrix. Now it is easy to check that the Lie algebra spanned by d_1, d_2, e_{12} and e_{13} is conjugate to \mathfrak{r} and so it makes sense to denote $[x, r_{1/3}]$ as $r(\mathfrak{r})$. Now applying Theorem 3.2, we get the boundary matrix

$$R(\mathfrak{r}) = \begin{pmatrix} 1 & -at & -2t & at & 2a^2t^2 & at^2 & 2t & at^2 & 2t^2 \\ 0 & 1 & 0 & 0 & -2at & -3t & 0 & -t & 0 \\ 0 & 0 & 1 & 0 & 0 & at & 0 & 0 & -t \\ 0 & 0 & 0 & 1 & 2at & t & 0 & 3t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -at & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which is clearly a quantization of $r(\mathfrak{r})$.

(iii) The two-parameter family of four-dimensional subalgebras of $\mathfrak{sl}(3)$

$$\mathfrak{q}_{a,b,c} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} + \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$

also carries classical r -matrices. For $(a, b, c) = (0, 1, -1)$ or $(1, 1, -2)$, there two classical r -matrices carried by $\mathfrak{q}_{a,b,c}$, for $a = c$ there are no classical r -matrices carried by $\mathfrak{q}_{a,b,c}$, and the remaining types each carry a single classical r . If $a \neq c$ then $r(\mathfrak{q}_{a,b,c}) = (ae_{11} + be_{22} + ce_{33}) \wedge e_{13} + (a - c)e_{12} \wedge e_{23}$ is a classical r -matrix with carrier $\mathfrak{q}_{a,b,c}$. We are unable to determine if this entire family consists of boundary r -matrices, but there is a distinguished one-parameter sub-family of $\mathfrak{q}_{a,b,c}$ consisting of boundary r -matrices. If $x = e_{13}$ then a simple computation shows that $\exp(t \text{ad } x) \cdot r_\lambda = r_\lambda + tr(\mathfrak{q}_{-1-\lambda, 2\lambda, 1-\lambda})$ and

thus $r(\mathfrak{q}_{-1-\lambda, 2\lambda, 1-\lambda})$ is a boundary r -matrix for every λ . The quantization of any $r(\mathfrak{q}_{a,b,c})$ is given by $\exp(2t\mathfrak{q}_{a,b,c})$ which is

$$R(\mathfrak{q}_{a,b,c}) = \begin{pmatrix} 1 & 0 & at & 0 & 0 & 0 & -at & 0 & -act^2 \\ 0 & 1 & 0 & 0 & 0 & at - ct & 0 & -bt & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -ct \\ 0 & 0 & 0 & 1 & 0 & bt & 0 & ct - at & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & ct \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the cases $r(\mathfrak{q}_{-1-\lambda, 2\lambda, 1-\lambda})$ Theorem 3.2 also applies and the resulting boundary R -matrix coincides with $R(\mathfrak{q}_{-1-\lambda, 2\lambda, 1-\lambda})$.

As mentioned earlier, $\mathfrak{q}_{0,1,-1}$ and $\mathfrak{q}_{1,1,-2}$ each carry an additional classical r . These are given by $r'(\mathfrak{q}_{0,1,-1}) = (e_{22} - e_{33}) \wedge e_{23} + e_{12} \wedge e_{13}$ and $r'(\mathfrak{q}_{1,1,-2}) = (e_{11} + e_{22} - 2e_{33} + e_{23}) \wedge e_{12} + 3e_{12} \wedge e_{23}$. We do not know whether these are boundary r -matrices but, fortunately, they are easy to quantize. Their exponentials, $\exp(2tr'(\mathfrak{q}_{0,1,-1}))$ and $\exp(2tr'(\mathfrak{q}_{1,1,-2}))$, satisfy the quantum Yang-Baxter equation. These are given by

$$R'(\mathfrak{q}_{0,1,-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t & 0 & -t & t^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t & 0 & -t & t^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$R'(\mathfrak{q}_{1,1,-2}) = \begin{pmatrix} 1 & 0 & t & 0 & 0 & 0 & -t & 0 & 2t^2 \\ 0 & 1 & 0 & 0 & 0 & 3t & 0 & -t & -t \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2t \\ 0 & 0 & 0 & 1 & 0 & t & 0 & -3t & t \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- (iv) The two dimensional non-abelian Lie algebra has three types of embeddings in $\mathfrak{sl}(3)$. Each embedding carries a unique classical r .

(a) For any scalar λ set

$$\mathfrak{b}_\lambda = \begin{pmatrix} -1 + \lambda & 0 & 0 \\ 0 & 1 + \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} + \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now $\exp(t \operatorname{ad} e_{12}) \cdot r_\lambda = r_\lambda + t((-1 + \lambda)e_{11} + (1 + \lambda)e_{22} - 2\lambda e_{33}) \wedge e_{12}$ and so $r(\mathfrak{b}_\lambda) = ((-1 + \lambda)e_{11} + (1 + \lambda)e_{22} - 2\lambda e_{33}) \wedge e_{12}$ is a boundary

classical r -matrix. Theorem 3.2 then gives the boundary R -matrix

$$R(\mathfrak{b}_\lambda) = \begin{pmatrix} 1 & \lambda t - t & 0 & t - \lambda t & -\lambda^2 t^2 + t^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -t - \lambda t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2\lambda t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t + \lambda t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2\lambda t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Let

$$\mathfrak{b}^{(0)} = * \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + * \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $r(\mathfrak{b}^{(0)}) = (e_{12} + e_{23}) \wedge (2e_{11} - 2e_{33})$. We are unable to determine if this is a boundary r -matrix. Moreover, $\exp(2t\mathfrak{b}^{(0)})$ does not satisfy the quantum Yang-Baxter equation. Theorem 4.1 however does apply here and specializing the universal R gives

$$R(\mathfrak{b}^{(0)}) = \begin{pmatrix} 1 & -2t & 2t^2 & 2t & 0 & 0 & 2t^2 & 0 & 0 \\ 0 & 1 & -2t & 0 & 0 & 2t^2 & 0 & 0 & -2t^3 \\ 0 & 0 & 1 & 0 & 0 & -2t & 0 & 0 & 2t^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2t & 2t^2 & 2t^3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2t & 2t^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) Let

$$\mathfrak{b}^{(1)} = * \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The classical r here is $r(\mathfrak{b}^{(1)}) = e_{13} \wedge (2e_{11} - e_{22} - e_{33} + e_{23})$ but just like case (b), we are unable to determine if this is a boundary r -matrix but specializing the universal R of Theorem 4.1 for this representation gives the R -matrix

$$R(\mathfrak{b}^{(1)}) = \begin{pmatrix} 1 & 0 & -2t & 0 & 0 & 0 & 2t & 0 & 2t^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -t & t \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 1 & 0 & t & 0 & 0 & -t \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark. In [1] and [6] it is claimed that there are four embeddings of the two-dimensional non-abelian Lie algebra in $\mathfrak{sl}(3)$. However, the algebra

of type $\mathfrak{b}^{(2)}$, in [1] is three-dimensional and the algebra of type $C_{1/3}^{1,1}$ in [6] is conjugate to type $\mathfrak{b}^{(c)}$ discussed in item (a).

- (v) There are three embeddings of the two-dimensional abelian Lie algebra in $\mathfrak{sl}(3)$.

(a) The Cartan subalgebra

$$\mathfrak{h} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

carries the unique classical r -matrix $r(\mathfrak{h}) = (e_{11} - e_{22}) \wedge (e_{22} - e_{33})$. It is a boundary r -matrix since $\lim_{\lambda \rightarrow \infty} (1/\lambda)r_\lambda = r(\mathfrak{h})$. The modified R -matrix which quantizes r_λ is $\exp(t\lambda r(\mathfrak{h})) \cdot \exp(2t\gamma) \cdot \exp(t\lambda r(\mathfrak{h}))$. Denote this modified R -matrix by Q . Then if $t = m/\lambda$, we have that $\lim_{\lambda \rightarrow \infty} Q = \exp(2mr(\mathfrak{h}))$ which, being a diagonal matrix, satisfies the quantum Yang-Baxter equation. Hence

$$R(\mathfrak{h}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-m} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-m} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-m} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is a boundary R -matrix.

- (b) The abelian subalgebra

$$\mathfrak{h}^{(1)} = * \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + * \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the carrier of $r(\mathfrak{h}^{(1)}) = (e_{11} + e_{22} - 2e_{33}) \wedge e_{12}$. It is a boundary r -matrix because if $r(\mathfrak{b}_\lambda)$ is the boundary r -matrix associated to the non-abelian Lie algebra \mathfrak{b}_λ , then $\lim_{\lambda \rightarrow \infty} (1/\lambda)r(\mathfrak{b}_\lambda) = r(\mathfrak{h}^{(1)})$ and so it is a limiting case of boundary classical r -matrices and so must be one itself. The same argument works on the quantum level, giving the boundary R -matrix

$$R(\mathfrak{h}^{(1)}) = \begin{pmatrix} 1 & t & 0 & -t & -t^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) The only other abelian subalgebras of $\mathfrak{sl}(3)$ are

$$\mathfrak{h}^{(\lambda,1)} = * \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Up to conjugacy however, there are only two algebras in this family, $\mathfrak{h}^{(1,1)}$ and $\mathfrak{h}^{(0,1)}$. In any case, the associated classical r -matrix is $r(\mathfrak{h}^{(\lambda,1)}) = (e_{12} + \lambda e_{23}) \wedge e_{13}$. This is also a limiting case of boundary solutions. Recall that the Lie algebra $\mathfrak{q}_{-1,0,1}$ is in the distinguished sub-family of $\mathfrak{q}_{a,b,c}$ which carries boundary solutions. Set $x = e_{12} + \lambda e_{13}$. Then $\exp(t \operatorname{ad} x) \cdot r(\mathfrak{q}_{-1,0,1}) = r(\mathfrak{q}_{-1,0,1}) - 2tr(\mathfrak{h}^{(\lambda,1)})$ and so $r(\mathfrak{h}^{(\lambda,1)})$ is a boundary r -matrix. The associated boundary R -matrix is found in the same way, specifically it is

$$R(\mathfrak{h}^{(\lambda,1)}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t & 0 & -t & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda t \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \lambda t \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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